Université de Bejaia **Introduction à la théorie ergodique.**
Département de mathématiques **2019-2020.**

Corrigé type examen final.

Exercice 1 (05.50 *pts*)

1. On considère $I = \left[0, \dfrac{\pi}{12}\right]$. On a :

$$\begin{cases} R_{\frac{\pi}{6}}(I) = \left[\dfrac{\pi}{6}, \dfrac{\pi}{6} + \dfrac{\pi}{12}\right] = \left[\dfrac{\pi}{6}, \dfrac{\pi}{4}\right] \\ R^2_{\frac{\pi}{6}}(I) = R_{\frac{\pi}{6}}\left(\left[\dfrac{\pi}{6}, \dfrac{\pi}{4}\right]\right) = \left[\dfrac{\pi}{6} + \dfrac{\pi}{6}, \dfrac{\pi}{4} + \dfrac{\pi}{6}\right] = \left[\frac{1}{3}\pi, \frac{5}{12}\pi\right] \\ R^3_{\frac{\pi}{6}}(I) = R_{\frac{\pi}{6}}\left(\left[\frac{1}{3}\pi, \frac{5}{12}\pi\right]\right) = \left[\frac{1}{3}\pi + \dfrac{\pi}{6}, \frac{5}{12}\pi + \dfrac{\pi}{6}\right] = \left[\frac{1}{2}\pi, \frac{7}{12}\pi\right] \\ R^4_{\frac{\pi}{6}}(I) = R_{\frac{\pi}{6}}\left(\left[\frac{1}{2}\pi, \frac{7}{12}\pi\right]\right) = \left[\frac{1}{2}\pi + \dfrac{\pi}{6}, \frac{7}{12}\pi + \dfrac{\pi}{6}\right] = \left[\frac{2}{3}\pi, \frac{3}{4}\pi\right] \\ R^5_{\frac{\pi}{6}}(I) = R_{\frac{\pi}{6}}\left(\left[\frac{2}{3}\pi, \frac{3}{4}\pi\right]\right) = \left[\frac{2}{3}\pi + \frac{\pi}{6}, \frac{3}{4}\pi + \frac{\pi}{6}\right] = \left[\frac{5}{6}\pi, \frac{11}{12}\pi\right] \\ R^6_{\frac{\pi}{6}}(I) = R_{\frac{\pi}{6}}\left(\left[\frac{5}{6}\pi, \frac{11}{12}\pi\right]\right) = \left[\frac{5}{6}\pi + \frac{\pi}{6}, \frac{11}{12}\pi + \frac{\pi}{6}\right] = \left[0, \dfrac{\pi}{12}\right] \end{cases}$$

Soit $E = \left[0, \dfrac{\pi}{12}\right] \cup \left[\frac{1}{3}\pi, \frac{5}{12}\pi\right] \cup \left[\frac{1}{2}\pi, \frac{7}{12}\pi\right] \cup \left[\frac{2}{3}\pi, \frac{3}{4}\pi\right] \cup \left[\frac{5}{6}\pi, \frac{11}{12}\pi\right]$.
On a $R_{\frac{\pi}{6}}(E) = E$ et $0 < \lambda(E) = \dfrac{\pi}{12} < 1$. (03.00 *pts*)

2.

$$\begin{cases} 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3} \leq y \leq \frac{2}{3} \end{cases} \Rightarrow \frac{1}{3} \leq x + y \leq 1 \Rightarrow \begin{cases} \frac{1}{3} \leq x + y < 1 \\ ou \\ x + y = 1 \end{cases}$$

$$\Rightarrow F\left(\left[0, \frac{1}{3}\right] \times \left[\frac{1}{3}, \frac{2}{3}\right]\right) = \left(\left[\frac{1}{3}, 1\right[\cup \{0\}\right) \times \left[\frac{1}{3}, \frac{2}{3}\right]$$

$$\begin{cases} \frac{2}{3} \leq x \leq 1 \\ \frac{1}{3} \leq y \leq \frac{2}{3} \end{cases} \Rightarrow \begin{cases} 1 \leq x + y \leq \frac{5}{3} \\ \frac{1}{3} \leq y \leq \frac{2}{3} \end{cases}$$

$$\Rightarrow F\left(\left[\frac{2}{3}, 1\right] \times \left[\frac{1}{3}, \frac{2}{3}\right]\right) = \left[0, \frac{2}{3}\right] \times \left[\frac{1}{3}, \frac{2}{3}\right]$$

Exercice 2 (10.00 pts)

1. (02.50 pts)

$$\begin{aligned} F\left(\left[\frac{1}{4},\frac{1}{2}\right]\right) &= F\left(\left[\frac{1}{4},\frac{1}{3}\right]\right)\cup F\left(\left[\frac{1}{3},\frac{1}{2}\right]\right)=\left[0,\frac{1}{4}\right]\cup\left[0,\frac{1}{2}\right]=\left[0,\frac{1}{2}\right]. \\ F^{-1}\left(\left[\frac{1}{4},\frac{1}{2}\right]\right) &= \left[\frac{1}{6},\frac{1}{4}\right]\cup\left[\frac{5}{12},\frac{1}{2}\right]\cup\left[\frac{5}{6},\frac{11}{12}\right]. \\ F^{-1}\left(\left[\frac{3}{4},1\right]\right) &= \left[0,\frac{1}{12}\right]\cup\left[\frac{7}{12},\frac{2}{3}\right]\cup\left[\frac{2}{3},\frac{3}{4}\right]=\left[0,\frac{1}{12}\right]\cup\left[\frac{7}{12},\frac{3}{4}\right] \end{aligned}$$

2.

$$\begin{cases} -3x+1=a \\ 3x-1=a \\ -3x+3=a \end{cases} \Rightarrow \begin{cases} x=\frac{1}{3}-\frac{1}{3}a \\ x=\frac{1}{3}a+\frac{1}{3} \\ x=1-\frac{1}{3}a \end{cases}$$

$$F^{-1}([a,b])=\left[\frac{1}{3}-\frac{1}{3}b,\frac{1}{3}-\frac{1}{3}a\right]\cup\left[\frac{1}{3}a+\frac{1}{3},\frac{1}{3}b+\frac{1}{3}\right]\cup\left[1-\frac{1}{3}b,1-\frac{1}{3}a\right].\ (01\ point\times 3)$$

$$\begin{aligned} \lambda\left(F^{-1}([a,b])\right) &= \lambda\left(\left[\frac{1}{3}-\frac{1}{3}b,\frac{1}{3}-\frac{1}{3}a\right]\cup\left[\frac{1}{3}a+\frac{1}{3},\frac{1}{3}b+\frac{1}{3}\right]\cup\left[1-\frac{1}{3}b,1-\frac{1}{3}a\right]\right) \\ &= \left(\frac{1}{3}-\frac{1}{3}a\right)-\left(\frac{1}{3}-\frac{1}{3}b\right)+\left(\frac{1}{3}b+\frac{1}{3}\right)-\left(\frac{1}{3}a+\frac{1}{3}\right)+\left(1-\frac{1}{3}a\right)-\Big(\\ &= b-a=\lambda([a,b]).\ (00.50\ pts) \end{aligned}$$

3.

$$\begin{aligned} \frac{1}{4} &\in \left[0,\frac{1}{3}\right]\Rightarrow\mu_{\frac{1}{4}}\left(\left[0,\frac{1}{3}\right]\right)=1.\ (00.50\ pts) \\ F\left(\frac{1}{4}\right) &\in F\left(\left[0,\frac{1}{3}\right]\right)\Rightarrow\frac{1}{4}\in F\left(\left[0,\frac{1}{3}\right]\right)\Rightarrow\mu_{\frac{1}{4}}\left(F\left(\left[0,\frac{1}{3}\right]\right)\right)=1.\ (00.75\ p \\ \text{Comme } F\left(\frac{1}{4}\right) &\in \left[0,\frac{1}{3}\right]\Rightarrow\frac{1}{4}\in F^{-1}\left(\left[0,\frac{1}{3}\right]\right)\Rightarrow\mu_{\frac{1}{4}}\left(F^{-1}\left(\left[0,\frac{1}{3}\right]\right)\right)=1.\ (00.75\ pt \end{aligned}$$

4. On a $\lambda\left(\left[0,\frac{1}{3}\right]\right)=\frac{1}{3}>0$, par le théorème de récurrence de Poincaré pour presque tout $x\in[0,1]$ il existe un entier n_0 tel que $F^{n_0}(x)\in\left[0,\frac{1}{3}\right]$. (02 pts)

Exercice 3 (04.50 *pts*)

1.

$$\begin{cases} \mu(1) = \frac{1}{4}\mu(2) + \frac{1}{4}\mu(3) \\ \mu(2) = \frac{1}{2}\mu(1) + \frac{1}{4}\mu(3) + \frac{1}{2}\mu(4) + \frac{1}{2}\mu(5) \\ \mu(3) = \frac{1}{2}\mu(1) + \frac{1}{4}\mu(2) + \frac{1}{2}\mu(6) + \frac{1}{2}\mu(7) \\ \mu(4) = \frac{1}{4}\mu(2) + \frac{1}{2}\mu(5) \\ \mu(5) = \frac{1}{4}\mu(2) + \frac{1}{2}\mu(4) \\ \mu(6) = \frac{1}{4}\mu(3) + \frac{1}{2}\mu(7) \\ \mu(7) = \frac{1}{4}\mu(3) + \frac{1}{2}\mu(6) \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} \mu(1) \\ \mu(2) \\ \mu(3) \\ \mu(4) \\ \mu(5) \\ \mu(6) \\ \mu(7) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}}_{T} \begin{pmatrix} \mu(1) \\ \mu(2) \\ \mu(3) \\ \mu(4) \\ \mu(5) \\ \mu(6) \\ \mu(7) \end{pmatrix} \text{ (01. } pts\text{)}$$

$$X^{(n+1)} = (0.3)\,.T.X^{(n)} + (0.7)\,E \text{ où } E_i = \frac{1}{7}, 1 \leq i \leq 7. \text{ (00.50 } pts\text{)}$$

2. Calculer la probabilité de se retrouver en page 6 au bout de 2 clics en étant en page 2. (01.50 *pts*)

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}^2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{16} & \frac{1}{16} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{7}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{7}{16} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{16} & \frac{3}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & \frac{3}{8} & 0 & 0 \\ \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & 0 & 0 & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{3}{8} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & 0 & 0 & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{16} \Rightarrow \mu(6) = \frac{1}{16}.$$

3. Sachant qu'on a des chances égales de se trouver sur une page quelconque, quelle est la probabilité de se trouver en page 6 au bout d'un clic. (01.50 *pts*)

$$\begin{pmatrix} 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \end{pmatrix} = \begin{pmatrix} \frac{1}{14} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{3}{28} \\ \frac{3}{28} \\ \frac{3}{28} \\ \frac{3}{28} \end{pmatrix} \Rightarrow \mu(6) = \frac{3}{28}$$