Université Ferhat Abbas, Sétif I

Département de mathématiques

Introduction aux Probabilités

§3. Variables aléatoires discrètes

1.

x	$x<2$	$-2\le x<-1$	$-1\le x<0$	$0\le x<1$	$1\le x<2$	$x\ge 2$
$F(x)$	0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{23}{40}$	$\frac{7}{10}$	1

$$E(X)=-2\left(\tfrac{1}{8}\right)-1\left(\tfrac{1}{4}\right)+0\left(\tfrac{1}{5}\right)+1\left(\tfrac{1}{8}\right)+2\left(\tfrac{3}{10}\right)=\frac{9}{40}$$

$$E\left(X^2\right)=(-2)^0\left(\tfrac{1}{8}\right)+(-1)^2\left(\tfrac{1}{4}\right)+0\left(\tfrac{1}{5}\right)+(1)^2\left(\tfrac{1}{8}\right)+(2)^2\left(\tfrac{3}{10}\right)=\frac{17}{10}$$

$$Var\left(X\right)=E\left(X^2\right)-\left[E\left(X\right)\right]^2=\frac{17}{10}-\left(\frac{9}{40}\right)^2=\frac{2639}{1600}=1.6494$$

2. (i)

x		2	3	11
$F(x)$	0	$\frac{1}{3}$	$\frac{5}{6}$	1

, $E\left(X\right)=4, E\left(X^2\right)=26, V(X)=10$

(ii)

x		-5	-4	1	2
$F(x)$	0	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{7}{8}$	1

, $E\left(X\right)=-1, E\left(X^2\right)=9.25, V(X)=8.25$

(iii)

x		-1	0	3
$F(x)$	0	$\frac{1}{3}$	$\frac{2}{3}$	1

,$p=\frac{1}{3}, E\left(X\right)=\frac{2}{3}, E\left(X^2\right)=\frac{10}{3}, V(X)=2.889$

3. (i)

x_i	2	4	6	8	10	12
$P\left(x_i\right)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

, $\mu_x=7, E\left(X^2\right)=\frac{364}{6}=60.7, \sigma_x^2=E\left(X^2\right)-\left(\mu_x\right)^2=11.7, \sigma_x=\sqrt{11.7}=3.4$

(ii)

y_i	1	3
$P\left(y_i\right)$	$\frac{1}{2}$	$\frac{1}{2}$

, $\mu_y=2, E\left(Y^2\right)=5, \sigma_x^2=E\left(X^2\right)-\left(\mu_x\right)^2=1, \sigma_x=1$

(iii)

z_i	3	7	11	15
$P\left(z_i\right)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

, $\mu_z=\mu_x+\mu_y=9, E\left(Z^2\right)=\frac{574}{6}=95.7, \sigma_z^2=E\left(Z^2\right)-\left(\mu_z\right)^2=14.7, \sigma_z=3.8$

(iv)

w_i	2	6	10	12	24	36
$P\left(w_i\right)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

, $\mu_W=15, E\left(W^2\right)=\frac{2156}{6}=359.3, \sigma_W^2=E\left(W^2\right)-\left(\mu_W\right)^2=134.3, \sigma_W=11.6$

4.

x_i	0	1	2	3	4	5
$P\left(w_i\right)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

, $E\left(X\right)=2.5, Var\left(X\right)=2.92$

y_i	2	3	4	5	6	7	8	9	10	11	12
$P\left(y_i\right)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$E\left(Y\right)=7, Var\left(Y\right)=5.83$

5. $P\left(X=i\right)=\frac{\binom{3}{i}\binom{9}{3-i}}{\binom{12}{3}}, i=0,1,2,3$

x_i	0	1	2	3
$P\left(x_i\right)$	$\frac{84}{220}$	$\frac{108}{220}$	$\frac{27}{220}$	$\frac{1}{220}$

,$\mu_X=\frac{108+54+3}{220}=\frac{3}{4}$

6.

x_i	1	2	-5
$P(z_i)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

, $\mu_X = \frac{1}{2} + \frac{2}{4} - \frac{5}{4} = -\frac{1}{4}$ donc le jeux est defavorable pour le joueur.

7.

x_i	0	1	2	3	4
$P(x_i)$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

, $P\left\{\frac{1}{2} < X < \frac{3}{2}\right\} = \frac{1}{10}$, $Var(X) = 1.96$

8.

x	$x < 1$	$1 \leq x < 3$	$3 \leq x < 4$	$4 \leq x < 6$	$6 \leq x < p$	$x \geq 9$
$F(x)$	0	2/10	4/10	5/10	7/10	1

9. $\sum_{k=1}^{\infty} P(X = k) = \sum_{k=1}^{\infty} \dfrac{c}{k^2} = c \sum_{k=1}^{\infty} \dfrac{1}{k^2} = c\frac{\pi^2}{} = 1 \Longrightarrow c = \frac{6}{\pi^2}$

   $E(X) = \sum_{k=1}^{\infty} kP(X = k) = \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k}$ qui la series harmonic qui diverge.

10.

x_i	1	2	3	4	5	6
$P(x_i)$	$\frac{1}{2}$	$\frac{5}{10}\frac{5}{9}$	$\frac{5}{10}\frac{4}{9}\frac{5}{8}$	$\frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{5}{7}$	$\frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{2}{7}\frac{5}{6}$	$\frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{2}{7}\frac{1}{6}$

11. $P(X = 2) = \binom{4}{2}\left(\frac{3}{7}\right)^2\left(\frac{4}{7}\right)^2 = 0.35985$

12. $P(X \geq 4) = P(X = 4) + P(X = 5) = \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1 + \binom{5}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^0 = 0.045267$

13. (a) $P(X = 2) = \binom{4}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = \frac{8}{27}$

    (b) $P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{4}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^4 = \frac{80}{81}$

    (c) $P(X > 2) = P(X = 3) + P(X = 4) = \binom{4}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^1 + \binom{4}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^0 = \frac{16}{27}$

14. (a) $X \sim B(10, 0.1)$

    (b) $E(X) = np = 10 \times 0.1 = 1$ et $Var(X) = npq = 10 \times 0.1 \times 0.9 = 0.9$

    (c) $P(X \leq 1) = P(X = 0) + P(X = 1) = \binom{10}{0}(0.1)^0(0.9)^{10} + \binom{10}{1}(0.1)^1(0.9)^9 = 0.7361$

    (d) Puisque $p = 0.1$ est petit on peut approximer $X = B(10, 0.1) \approx Y = Poi(1)$.
    $P(X \leq 1) \approx P(Y \leq 1) = P(Y = 0) + P(Y = 1) = e^{-1}\frac{1^0}{0!} + e^{-1}\frac{1}{1!} = 2e^{-1} \approx 0.7358$

15. (a) $P(X \geq 3) = 1 - P(X \leq 2) = 1 - e^{-3}\frac{3^0}{0!} - e^{-3}\frac{3^1}{1!} - e^{-3}\frac{3^2}{2!} = 1 - \frac{17}{2}e^{-3} \approx 0.5768$

    (b) $P(X \geq 3 | X \geq 1) = \frac{P(X \geq 3)}{P(X \geq 1)} = \frac{1 - \frac{17}{2}e^{-3}}{1 - e^{-3}} = 0.6070$

16. X est une Bernoulli avec $p = \frac{1}{3}$. $E(X) = p = \frac{1}{3}$ et $\mathrm{Var}(X) = pq = \frac{2}{9}$.

17. (a) $X = B\left(7, \frac{1}{4}\right)$, $P(X \geq 2) = 1 - P(X \leq 1) = 1 - P(X = 0) - P(X = 1) = 1 - \binom{7}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^7 - \binom{7}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^6 = \frac{4547}{8192} \approx 0.5550$

    (b) $P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{n}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^n = 1 - \left(\frac{3}{4}\right)^n \geq \frac{2}{3} \Leftrightarrow n \geq \frac{-\ln 3}{\ln 3 - \ln 4} \approx 4$

18. Soit $X \sim \mathcal{B}(n, p)$ une variable aléatoire suivant la loi binômiale. Demontrer que $E(X) = np$ et $Var(X) = npq$.

$$
\begin{aligned}
M_X(t) &\quad E\left(e^{tX}\right) = \textstyle\sum_{k=0}^{n} e^{tk}\binom{n}{k}p^k q^{n-k} = \sum_{k=0}^{n}\binom{n}{k}\left(pe^t\right)^k q^{n-k} = \left(pe^t + q\right)^n \\
M_X'(t) &= npe^t\left(pe^t + q\right)^{n-1} \text{ et } M_X'(0) = E(X) = np \\
M_X''(t) &= n(n-1)\left(pe^t\right)^2\left(pe^t + q\right)^{n-2} + npe^t\left(pe^t + q\right)^{n-1} \text{ et} \\
M_X''(0) &= E\left(X^2\right) = n^2p^2 - np^2 + np \\
\mathrm{Var}(X) &= E\left(X^2\right) - [E(X)]^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq
\end{aligned}
$$

19. $X = B\left(8, \frac{1}{2}\right), \mu = np = 8 \times \frac{1}{2}4, P(X=4) = \binom{8}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^4 = \frac{35}{128}$

20. $X = B\left(10, \frac{1}{2}\right), P(X \geq 7) = \sum_{k=7}^{10}\binom{10}{k}\left(\frac{1}{2}\right)^{10} = \frac{11}{64} = 0.1719$

21. $X = B(1000, 0.0014)$,
$P(X \geq 2) = 1 - P(X \leq 1) = 1 - \binom{1000}{0}(0.0014)^0(0.9986)^{1000} - \binom{1000}{1}(0.0014)^1(0.9986)^{999} = 0.4083$

22. (a) 1 per / min $\Longrightarrow$ 2.5 per / 5 min donc $X = Poi(2.5)$, $P(X=0) = e^{-2.5} = 0.0821$

(b) $P(X \geq 4) = 1 - P(X \leq 3) = 1 - e^{-2.5}\left(1 + 2.5 + \frac{(2.5)^2}{2} + \frac{(2.5)^3}{6}\right) = 0.2424$

23. $X = \mathcal{B}(5, 0.3)$

(a) $P(X=0) = \binom{5}{0}(0.3)^0(0.7)^5 = 0.1681$

(b) $P(X=5) = \binom{5}{5}(0.3)^5(0.7)^0 = 0.0024$

(c) $\mu = np = 5 \times 0.3 = 1.5$

24. Soit $X \sim \mathcal{G}(p)$, $P(X=k) = pq^{k-1}, k = 1, 2, 3, \ldots$

(a) $P(X > n) = \sum_{k=n+1}^{\infty} pq^{k-1} = p\frac{q^n}{1-q} = q^n = (1-p)^n$.

(b) $F(n) = P(X \leq n) = 1 - P(X > n) = 1 - (1-p)^n$.

(c) $P(X > n+m | X > m) = \dfrac{P(X > n+m)}{P(X > m)} = \dfrac{(1-p)^{n+m}}{(1-p)^m} = (1-p)^n = P(X > n)$.

25. $M_X(t) = E\left(e^{tX}\right) = \sum_{k=0}^{\infty} e^{tk}e^{-\lambda}\dfrac{\lambda^k}{k!} = \sum_{k=0}^{\infty} e^{-\lambda}\dfrac{(\lambda e^t)^k}{k!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda\left(e^t-1\right)}$

$M'_X(t) = \lambda e^t e^{\lambda\left(e^t-1\right)} = \lambda e^{\lambda\left(e^t-1\right)+t}$, et $M'_X(0) = \lambda$.

$M''_X(t) = \lambda e^{\lambda\left(e^t-1\right)+t}(\lambda e^t + 1)$, et $M''_X(0) = \lambda(\lambda+1)$.

$Var(X) = E(X^2) - [E(X)]^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$.

26. Soit $X_1, X_2, \ldots, X_n$ des v.a.i. et $X_i \sim \mathcal{P}(\lambda_i)$ (poisson avec paramètre λ_i) et $X = \sum\limits_{i=1}^{n} X_i$.
Trouver la fonction génératrice de moment de X. Quelle est la loi de probabilité de X?
$M_X(t) = E\left(e^{tX}\right) = E\left(e^{t(X_1+X_2+\cdots+X_n)}\right) = E\left(e^{tX_1}\right)E\left(e^{tX_2}\right)\cdots E\left(e^{tX_n}\right)$ parce que les variables sont indépendantes. Donc on aura
$M_X(t) = e^{\lambda_1\left(e^t-1\right)}e^{\lambda_2\left(e^t-1\right)}\ldots e^{\lambda_n\left(e^t-1\right)} = e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)\left(e^t-1\right)} = e^{(\lambda)\left(e^t-1\right)}$
ou $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. On voit que fonction génératrice des moments de X est celle d'une poisson avec paramètre $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

27. Considérons la fonction $f(\lambda) = e^{-\lambda}\dfrac{\lambda^k}{k!}, \lambda > 0$.

$f'(\lambda) = -e^{-\lambda}\dfrac{\lambda^k}{k!} + e^{-\lambda}\dfrac{k\lambda^{k-1}}{k!} = \dfrac{e^{-\lambda}}{k!}\lambda^{k-1}(-\lambda + k) = 0$ quand $\lambda = k$. $f' > 0$ quand $\lambda < k$ et $f' < 0$ quand $\lambda > k$ qui nous donne un maximum quand $\lambda = k$.